Contextuality and the Kochen-Specker Theorem

Interpretations of Quantum Mechanics

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Interpretations of quantum mechanics

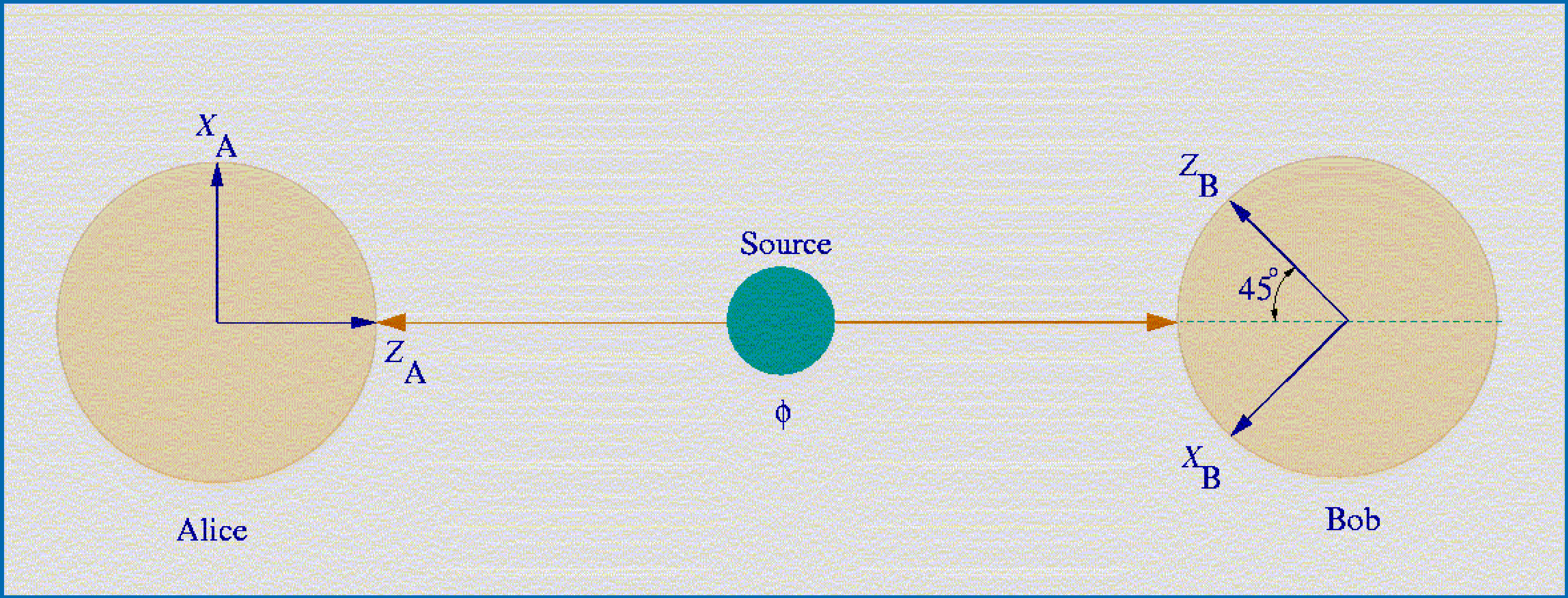
- **Copenhagen interpretation**
  the wavefunction has no reality
  probability is an essential part of nature
  (wavefunction collapse)

- **Hidden variable theory**
  quantum mechanics isn't complete
  hidden parameters needed for determinism
  must be non-local and contextual
  Bohmian mechanics

Comparison

Interpretation	Realism	Localism	Determinism	Unique History	Observer conscience
Copenhagen	No	Yes	No	Yes	No
Hidden Variables	Yes	No	Yes	Yes	No
Many Worlds	Yes	Yes	Yes	No	No
Many Minds	Yes	Yes	Yes	No	Yes
CCC	Yes	Yes	No	Yes	Yes

Einstein-Podolsky-Rosen paradox



Alice and Bob perform spin measurements of entangled particles

quantum mechanics + realism + locality + completeness

➔ ?“spooky” action at distance?

Bell's theorem

- "No physical theory of local hidden variables can ever reproduce all the predictions of quantum mechanics"

Original version: $1 + C(b,c) \geq \left| C(a,b) - C(a,c) \right|$

CHSH version: $-2 \leq E(a,b) - E(a,b') + E(a',b) + E(a',b') \leq +2$

Possible loopholes: detector loophole, locality loophole

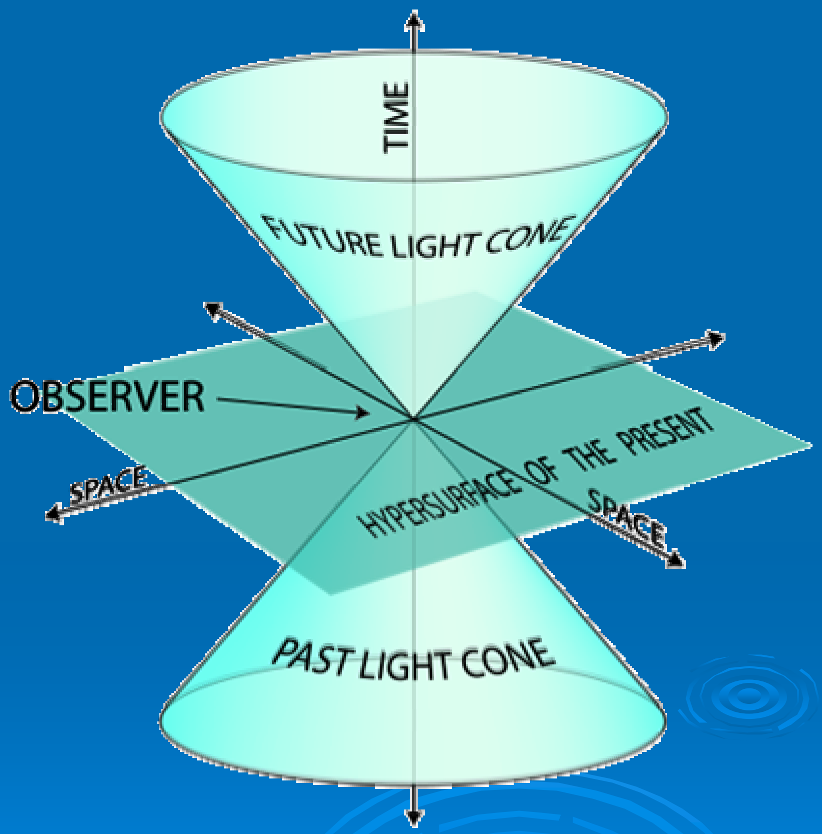
Locality

- Interaction limited to the immediate surroundings

- Causal contact

- No superluminal signals

- Bell's theorem



Contextuality

- results depend on the context of the experiment
- Take 2 sets of mutually commuting operator: A,B,C,… and A,L,M,…
- Not all B,C,… commute with all L,M,…
- If A measured with the first set is equal to A measured with the other set → non-contextuality
- Else contextuality

- Non-contextuality is more general than locality

- because: results must be independent for commuting observables even without spacelike separation

- Hidden variables: non-contextual or contextual?

→ Kochen-Specker theorem

The Kochen-Specker theorem

- “There is no non-contextual model with hidden variables in quantum mechanics”

the way to develop a no-go theorem for hidden variables

1932 - John von Neumann
1957 - A.M. Gleason
1966 - John S. Bell
1967 - Simon Kochen and Ernst Specker

mathematical formulation

- Let H be a Hilbert space of quantum mechanical state vectors of dimension $x \geq 3$. There is a set M of observables on H, containing y elements, such that the following two assumptions are contradictory:
- All y members of M simultaneously have values, i.e. are unambiguously mapped onto real numbers (designated, for observables A, B, C, …, by $v(A)$, $v(B)$, $v(C)$, …).
- Values of observables conform to the following constraints:
- If A, B, C are all compatible and $C = A+B$, then $v(C) = v(A)+v(B)$;
- If A, B, C are all compatible and $C = A \cdot B$, then $v(C) = v(A) \cdot v(B)$.

Derivation

- 3D state space of observables
  – angular momentum components of spin
- eigenvalue are 0 or 1
- square spin components in 3 orthogonal directions

$$S_u^2 + S_v^2 + S_w^2 = s(s+1) = 2$$

- they are mutually commuting
- can be simultaneous measured
- looking for a set of 3D vectors obeying conditions:

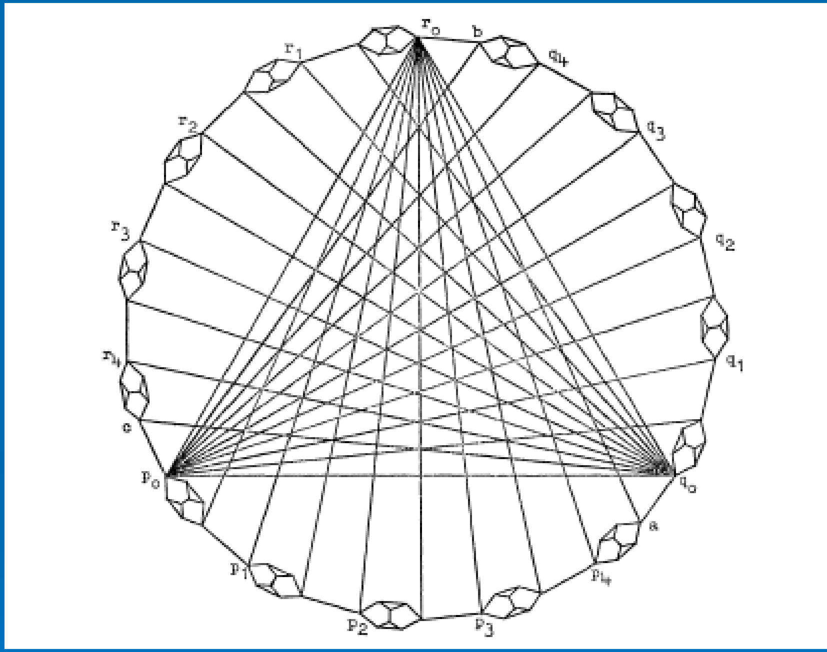
  red=1 blue=0

  It shall be impossible to colour each vector red or blue by using a subset containing one blue and two red vectors.

- doing some geometry …

- if the angle between 2 different coloured vectors <arctan(0.5), we can find uncolourable directions

- Kochen and Specker discovered 117 vectors



- also sets with fewer vectors possible

- it gets simpler in 4 dimensions

- now: observables are Pauli matrices for two independent spin ½-particles σ^1_μ and σ^2_ν

- using properties of Pauli matrices

- Observables are mutually commuting

- Product in each row is +1

- Product in first 2 columns is +1
  in the last column it is -1

σ_x^1	σ_x^2	$\sigma_x^1\sigma_x^2$
σ_y^2	σ_y^1	$\sigma_y^1\sigma_y^2$
$\sigma_x^1\sigma_y^2$	$\sigma_x^2\sigma_y^1$	$\sigma_z^1\sigma_z^2$

- These conditions can't be satisfied - contradiction!

Single photon experiment

- Proposed by Christoph Simon, Marek Zukowski, Harald Weinfurther and Anton Zeilinger in 2000
- Performed by Yun-Feng Huang, Chuan-Feng Li, Yong-Sheng Zhang, Jian-Wei Pan and Guang-Can Guo in 2002
- Uses photons polarisation and path degrees of freedom

- 4 observables Z_1, X_1, Z_2, X_2

- Predetermined non-contextual values: $v(Z_1)$, $v(X_1)$, $v(Z_2)$, $v(X_2)$

- assuming non-contextuality in our calculations and a suitable ensemble where:

$$v(Z_1) = v(Z_2) \qquad and \qquad v(X_1) = v(X_2)$$

- we get following relations:

$$v(Z_1).v(Z_2) = v(X_1).v(X_2) = 1$$

$$\Rightarrow v(Z_1).v(X_2) = v(X_1).v(Z_2)$$

$$v(Z_1.X_2) = v(Z_1).v(X_2)$$

- Now: quantum mechanical system of 2 qubits

- Observables: $Z_1 = \sigma_Z^{(1)}$ $X_1 = \sigma_X^{(1)}$

  $Z_2 = \sigma_Z^{(2)}$ $X_2 = \sigma_X^{(2)}$

- Use a joint eigenstate of the commuting product observables Z_1Z_2 and X_1X_2:

$$\left|\psi_1\right\rangle = \frac{1}{\sqrt{2}}\left(\left|+z\right\rangle\left|+z\right\rangle + \left|-z\right\rangle\left|-z\right\rangle\right) = \frac{1}{\sqrt{2}}\left(\left|+x\right\rangle\left|+x\right\rangle + \left|-x\right\rangle\left|-x\right\rangle\right)$$

- Quantum mechanics predicts: the value measured for Z_1X_2 will always be opposite to that of X_1Z_2

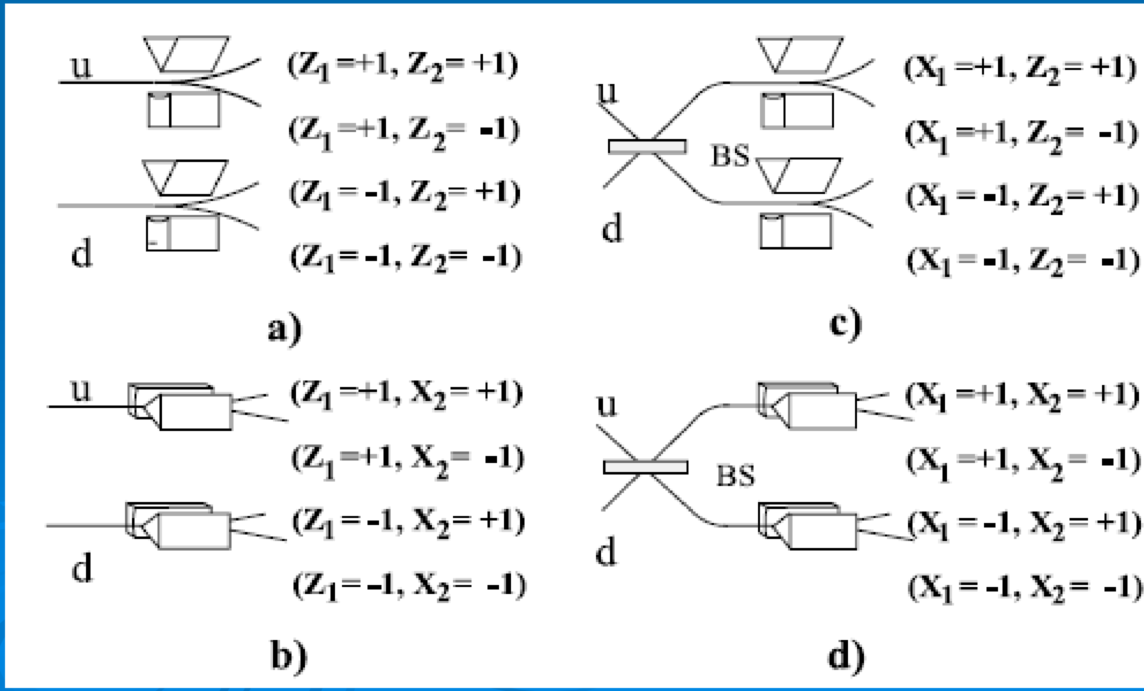
The state in the experiment: $\left|\psi_1\right\rangle = \frac{1}{\sqrt{2}}\left(\left|u\right\rangle\left|z+\right\rangle + \left|d\right\rangle\left|z-\right\rangle\right)$

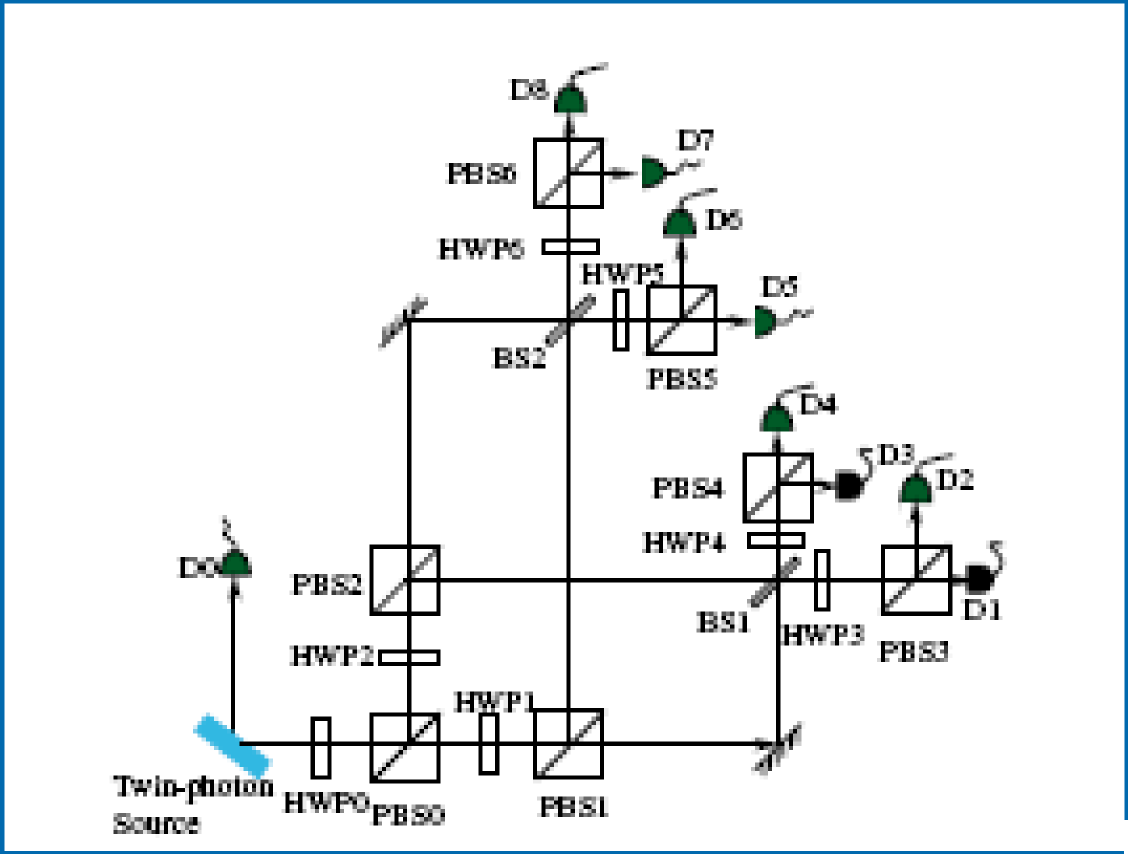
Observables:

$Z_1 = \left|u\right\rangle\left\langle u\right| - \left|d\right\rangle\left\langle d\right|$ $\qquad X_1 = \left|u'\right\rangle\left\langle u'\right| - \left|d'\right\rangle\left\langle d'\right|$

$Z_2 = \left|z+\right\rangle\left\langle z+\right| - \left|z-\right\rangle\left\langle z-\right|$ $\qquad X_2 = \left|x+\right\rangle\left\langle x+\right| - \left|x-\right\rangle\left\langle x-\right|$

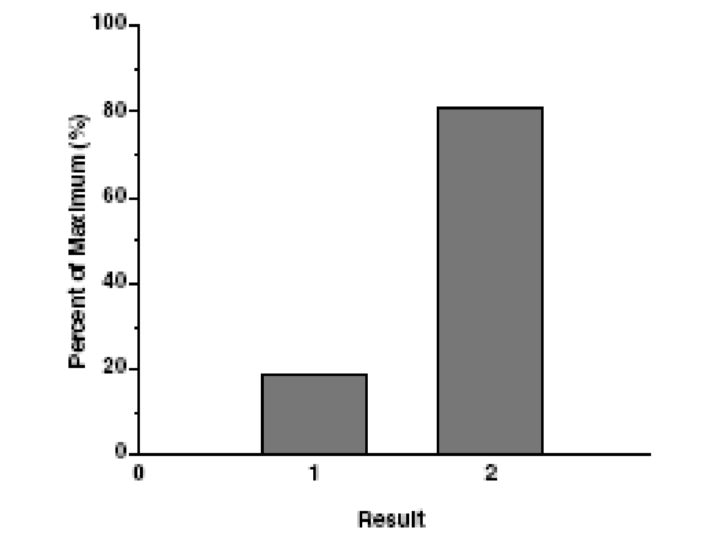
Measurement of these observables with beam-splitters and Stern-Gerlach devices





Experimental setup

Results show an agreement with quantum mechanics



Neutron optical experiment

- Performed by Yuji Hasegawa, Rudolf Loidl, Gerald Badurek, Matthias Baron and Helmut Rauch in 2003

- Neutron in a non-factorizable state and joint measurement of two commuting observables

- Degrees of freedom: path in interferometer
  spin states

- Total wavefunction $|\psi\rangle = \frac{1}{\sqrt{2}}(|\downarrow\rangle \otimes |p_1\rangle + |\uparrow\rangle \otimes |p_2\rangle)$

- Projections along orthogonal states

$$P^{s}_{\alpha;\pm 1} = \frac{1}{2}(|\uparrow\rangle \pm e^{i\alpha}|\downarrow\rangle)(\langle\uparrow| \pm e^{-i\alpha}\langle\downarrow|)$$

$$P^{p}_{\chi;\pm 1} = \frac{1}{2}(|p_1\rangle \pm e^{i\chi}|p_2\rangle)(\langle p_1| \pm e^{-i\chi}\langle p_2|)$$

- Projection operators realised by spin rotator and phase shifters in the experiment

- Expectation value of joint measurement of spin and path:

$$E(\alpha,\chi) = \langle\psi|P^{s}(\alpha)P^{p}(\chi)|\psi\rangle = \langle\psi|[(+1)P^{s}{}_{\alpha;+1} + (-1)P^{s}{}_{\alpha;-1}].[(+1)P^{p}{}_{\chi;+1} + (-1)P^{p}{}_{\chi;-1}]|\psi\rangle$$

- Using CHSH-inequality

$$-2 \leq S \leq +2$$

$$S = E(\alpha_1, \chi_1) - E(\alpha_1, \chi_2) + E(\alpha_2, \chi_1) + E(\alpha_2, \chi_2)$$

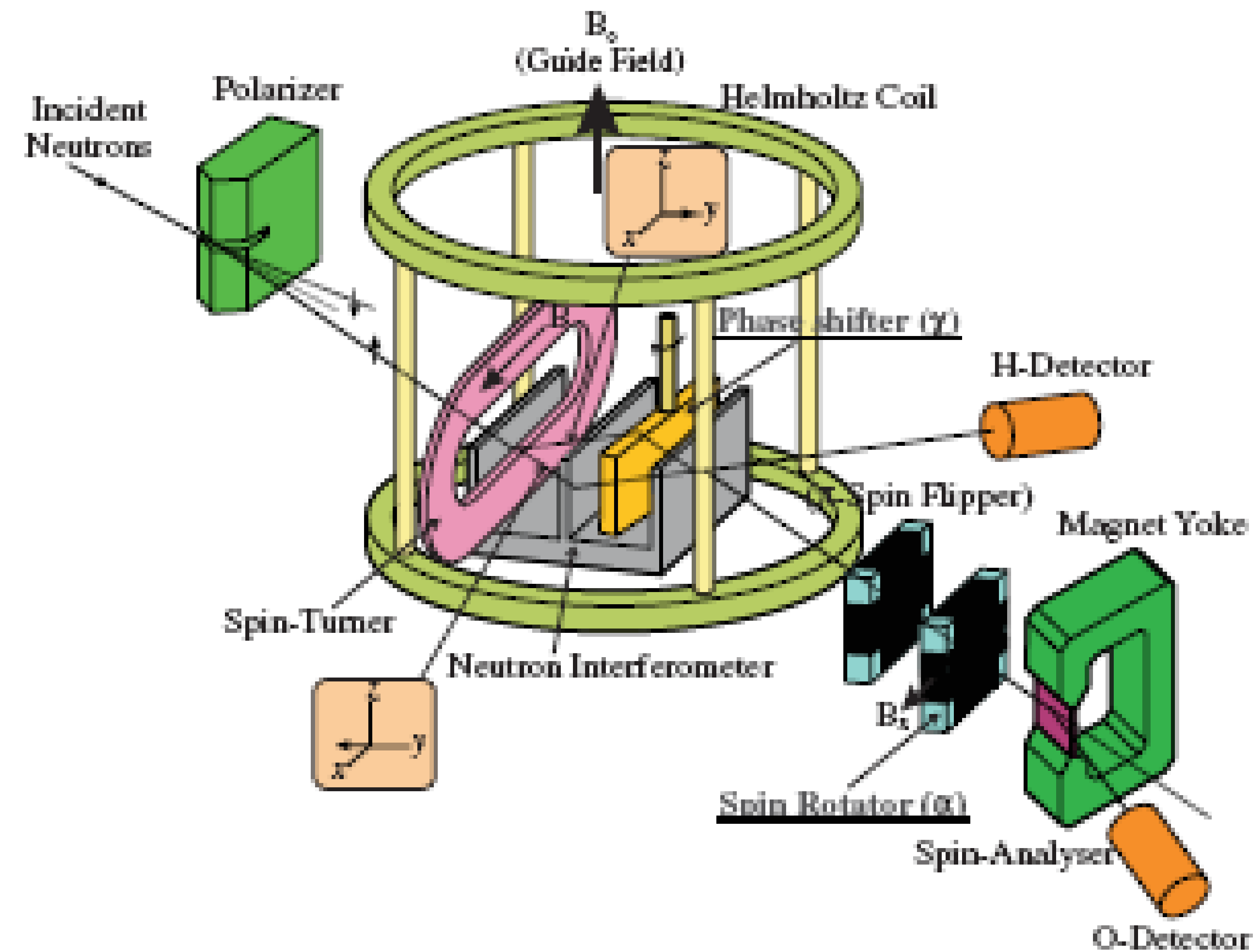
- Coincidence count rates + Quantum mechanics ➔

- Behaviour the expectation value

$$E(\alpha, \chi) = \cos(\alpha + \chi)$$

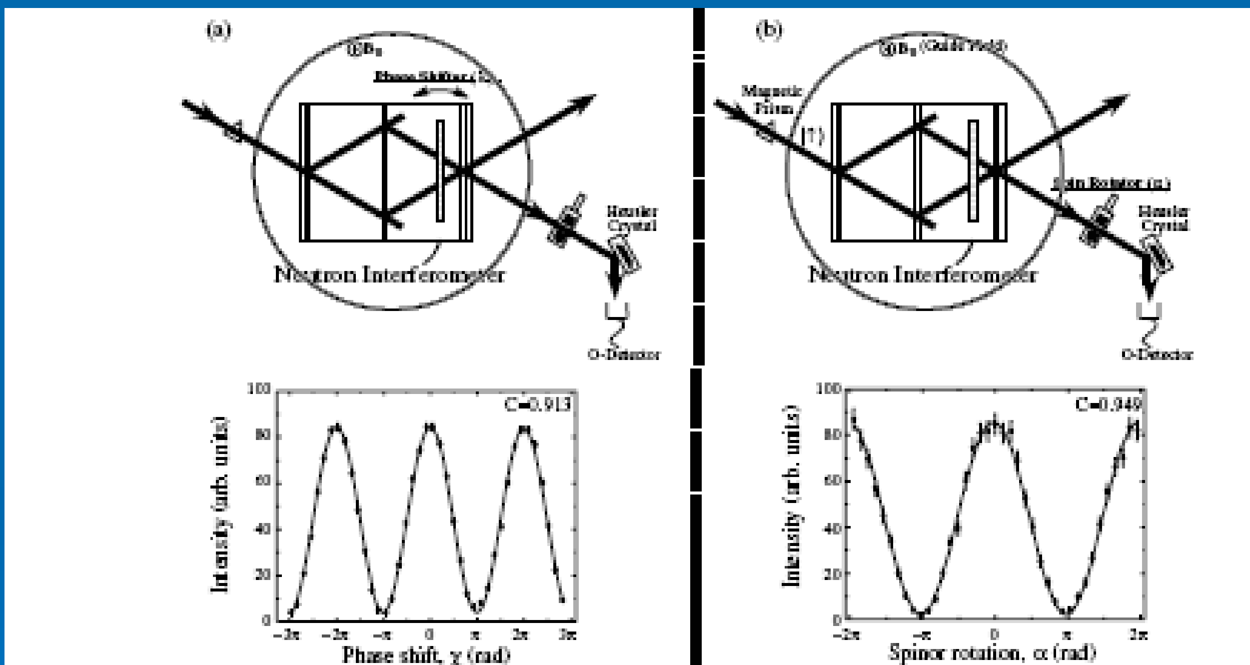
- theoretical maximum violation at some angles
  ➔ S ≈ 2.82

Experimental setup



Highly efficient detectors, but losses in interferometer
→ fair sampling theorem required

Correct behaviour of expectation value



Measured S=2.051±0.019 violets inequality

Summary

- Interpretations of quantum mechanics: Copenhagen Interpretation, hidden variables, many worlds, …
- EPR-Paradox → is quantum mechanics incomplete
- Bell's inequality: no local hidden variables
- Non-contextuality is more general than locality

- Kochen-Specker theorem:

  “There is no non-contextual model with hidden variables in quantum mechanics”

- successfully experimentally proved:

  Single photon and neutron optical experiments

- Only contextual hidden variables possible like Bohmian mechanics, but deeper insight?

- “Everything should be made as simple as possible, **but not simpler**.”

Any Questions?